# FURTHER DYNAMIC CO-ORDINATIONS.

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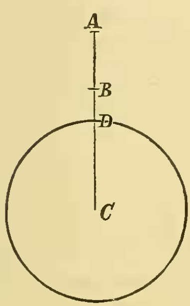
A further extension may be given to my co-ordination of the great natural forces, by means of the thermodynamic relations which subsist between constancy of pressure and constancy of volume.

In central forces, varying inversely as the square of the distance, a perpetual oscillation through a linear ellipse AC, with foci at the centre of a circle and at $2r$, would be synchronous with a perpetual revolution around the circle. The complete linear-elliptical orbit being $= 2d$, the mean velocity of linear oscillation, or the velocity of constant mean gaseous pressure $= \frac{2}{\pi}$ of the velocity of revolution; a velocity which would be attained, both in the centripetal and in the centrifugal phase of the oscillation, at $1.4232r$ $\left[= \frac{2\pi^2 r}{\pi^2 + 4}\right]$. The ratio of heat under constant volume to heat under constant pressure, as experimentally determined, is 1 : 1.421.*




Let $\varrho$ = radius of a gaseous nucleus which is sufficiently condensed to allow of chemical combinations, or the radius of constant volume; $r$ = radius of constant mean pressure. The *vis viva* of free revolution in a circular orbit varying inversely as radius, the ratio of the mean nucleal and atmospheric forces may be represented by the proportion

$$\varrho : r :: 1 : 1.4232$$

In elastic media, as the distances from the centre increase in arithmetical progression, the densities decrease, in geometrical progression if the central force is constant, in harmonic progression if the central force varies according to the law of inverse squares. Whatever may have been the beginnings of cosmo-taxis, whether through nebular condensation, meteoric accumulation, explosive rupture, or other unknown process, the secular mean actions and reactions between opposing forces should lead to similar numerical and harmonic results. In the language of Herschel,† "Among a crowd of solid bodies of whatever size, animated by independent and partially opposing impulses, motions opposite to each other *must* produce collision, destruction of velocity, and sub-

* Tyndall, *Heat a Mode of Motion*, 4th Ed., Sect. 74.

† *Outlines of Astronomy*, Sect. 872.

sidence or mean approach towards the centre of preponderant attraction; while those which conspire, or which remain outstanding after such conflicts, *must* ultimately give rise to circulation of a permanent character."

In the earliest stages of nucleal aggregation, when the primitive oscillating velocity subjects all particles to nearly equal impulses from every direction, but with a slight preponderance towards special nucleal centres, the variation from constancy of force may be so slight as to introduce a geometrical progression based on the above thermal ratio, 1 : 1.4232. Since the *nucleal* radius of a Sun which would rotate synchronously with planetary revolution varies as $\sqrt{t}$, while the planetary radius-vector, or radius of *possible nebular atmosphere*, varies as $(t)^{\frac{2}{3}}$, the atmospheric radius varies as (nucleal radius)$^{\frac{4}{3}}$. We have thus a basis for the geometrical series, $r, r^{\frac{4}{3}}, r^{\frac{5}{3}} \ldots r^{n}$. Now $1.4232^{\frac{4}{3}} = 1.6009$, or almost precisely the fundamental radius (1.6007) which Professor Alexander has pointed out in the arrangement of the Jovian system.*

It is also very nearly represented in the ratios between the nucleal radii of the inner pairs of planets, of the two principal planetary belts; ♀ : ☿ = 1.598; ♄ : ♃ = 1.576.† The next term of the series is $1.4232^{\frac{5}{3}} = 1.8008$, which is remarkably coincident, both with Professor Alexander's fundamental ratio‡ for the solar system (1 : 1.8), and with the "centre of explosive oscillation," or the linear centre of oscillation between a primitive centre of oscillation and a linear centre of gravity ($\frac{5}{9}$ and $\frac{4}{9}$). If the involution is carried to the fifth quadrangular pyramidal number, $1.4232^{35} = 231390$, which is within less than three per cent. of the half-modulus of light at Sun's surface, measured in solar radii. The pyramidal exponent, 35, is also within less than three per cent. of the possible solar atmosphere measured in solar radii; within less than five per cent. of the half-modulus of light measured in Neptunian vector-radii; and within less than three per cent. of the nucleal radius of a nebulous Sun which would rotate in a year of Uranus.

If these accordances are dependent upon the mutual interactions of the five principal masses in our system (⊙, ♃, ♄, ♅, ♆), we may reasonably look for still further accordances between the products of masses, which enter as factors into expressions of joint gravitating action. We accordingly find the following equation between the triangular powers of planetary masses, designating mean perihelion, mean, and mean aphelion, by subscript figures 1, 2, 3, respectively: $\left[\frac{\text{♆}_3}{\text{♄}_2}\right]^1 \times \left[\frac{\text{♅}_3}{\text{♄}_2}\right]^3 \times \left[\frac{\text{♃}_3}{\text{♄}_2}\right]^6 = 1$;

* *Statement and Exposition of Certain Harmonies in the Solar System* (Smithsonian Contributions, 280), p. 15.

† The simple ratio, 1.4232 is approximated in the nucleal radii of the outer pairs; ♂ ÷ ⊕ = 1.372 : ♆ ÷ ♅ = 1.4.

‡ *Op. cit.*, p. 4. I am informed by Prof. Alexander that he announced this ratio before the American Association in 1857.

or, if we introduce all of the five great masses:

$$\left[\frac{♆^3}{\odot}\right]^1 \times \left[\frac{♅^3}{\odot}\right]^3 \times \left[\frac{♃^3}{\odot}\right]^6 = \left[\frac{♄^2}{\odot}\right]^1 \times \left[\frac{♄^2}{\odot}\right]^3 \times \left[\frac{♄^2}{\odot}\right]^6$$

There is still so much uncertainty as to the masses of Neptune and Uranus, that it is impossible to tell how close this agreement may be, but the deviation from precise accuracy cannot be large. According to Newcomb's latest determinations of those masses, the equation gives two values for Saturn, one of which is slightly larger, the other slightly smaller, than Bessel's value. By looking a little further we may find relations which can be measured with greater certainty, and are therefore more satisfactory.

La Place found that if the mass of each planet be multiplied by the product of the square of the eccentricity and the square root of the mean distance, the sum of all the products will always retain the same magnitude ; also, that if each of the masses be multiplied by the product of the square of the orbital inclination and the square root of the mean distance, the sum of the products will always remain invariable. Now the square root of the mean distance varies inversely as the velocity of circular revolution at the mean distance, or inversely as the square root of the velocity of nucleal rotation at the same distance. It is therefore probable that the primitive undulations may have influenced the relative positions as well as the relative masses of the principal planetary orbs. Stockwell has found* the following relations :

I. *The mean motion of Jupiter's perihelion is exactly equal to the mean motion of the perihelion of Uranus, and the mean longitudes of these perihelia differ by exactly* 180°. II. *The mean motion of Jupiter's node on the invariable plane is exactly equal to that of Saturn, and the mean longitudes of these nodes differ by exactly* 180°.

I have already had frequent occasion to refer to the position of the nebular centre of planetary inertia $\left(\sqrt{\Sigma mr^2 \div \Sigma m}\right)$ in Saturn's orbit. If the four great planets were ranged in a line, Jupiter on one side of the Sun and the other planets on the other, the tidal influences, when Saturn was in mean position, would drive Jupiter, Uranus, and Neptune to, or towards, their respective aphelia. Those positions would accord with Stockwell's two theorems, they would approximate the centre of inertia very closely to Saturn's mean radius vector, and they would make the equation of the products of triangular powers applicable to vector radii, as well as to masses. For the logarithms of mean vector-radii of the four outer planets, according to Stockwell,† are :—

| | | | |
|---|---|---|---|
| Neptune, *mean Aphelion*, | 1.481951 | Neptune, | 1.481951 |
| Uranus, " | 1.301989 | $(\text{Uranus})^3$ | 3.905967 |
| Jupiter, " | .734588 | $(\text{Jupiter})^6$ | 4.407528 |
| | | 10) | 9.795446 |
| Saturn, *mean*, | .979496 | Saturn, | .979545 |

* *Memoir on the Secular Variations of the Elements of the Orbits of the Eight Principal Planets* (Smithsonian Contributions, 232), p. xiv

† *Ibid.* pp. 5, 38.

The difference between the actual value of log. ♄ *r. vec.*, and the value as found by the foregoing equation, $♆ \times ♅^{3} \times ♃^{6} = ♄^{10}$, is therefore, only .000049, representing a numerical difference of only $\frac{1}{80}$ of one per cent.

When the hypothetical nebular condensation had proceeded so far as to show the controlling planetary influence of Jupiter's mass, the mean perihelia of Saturn and Uranus were so fixed as to establish the following relationships of harmonic powers :

$$\left[\frac{♆_3}{♅_1}\right]^{\frac{1}{3}} \times \left[\frac{♅_3}{♄_1}\right]^{\frac{1}{2}} \times \left[\frac{♃_3}{♄_2}\right]^{\frac{1}{1}} = 1; \quad \left[\frac{♆_2}{♅_1}\right]^{\frac{1}{3}} \times \left[\frac{♅_3}{♄_1}\right]^{\frac{1}{2}} \times \left[\frac{♃_2}{♄_1}\right]^{\frac{1}{1}} = 1$$

Stockwell's logarithmic values are :

| | | | | | | |
|---|---|---|---|---|---|---|
| Neptune, | $♆_3$ | 1.481951 | $\alpha$ | $♆_1$ | 1.473327 | $\alpha^1$ |
| Uranus, | $♅_1$ | 1.262996 | $\beta$ | | | |
| " | $♅_3$ | 1.301989 | $\gamma$ | | | |
| Saturn, | $♄_1$ | .957973 | $\delta$ | $♄_2$ | .979496 | $\delta^1$ |
| Jupiter, | $♃_3$ | .734588 | $\varepsilon$ | $♃_2$ | .716237 | $\varepsilon^1$ |

$\frac{1}{3}(\alpha\text{-}\beta) + \frac{1}{2}(\gamma\text{-}\delta) + (\varepsilon\text{-}\delta^1) = .000085 =$ log. 1.0002

$\frac{1}{3}(\alpha^1\text{-}\beta) + \frac{1}{2}(\gamma\text{-}\delta) + (\varepsilon^1\text{-}\delta) = .000382 =$ " 1.0009

The theoretical differ from the actual values, by less than $\frac{1}{50}$ of one per cent. in the first, and by less than $\frac{1}{11}$ of one per cent. in the second equation.

At the hypothetical limit of struggle between opposing forces, which we are now considering, I have shown that the ratio between the velocity of incipient dissociation and the velocity of incipient aggregation is 1 : $\pi$. This ratio is found to prevail in a comparison of the vector-radii of the aphelion planets in each of the aphelion or supra-asteroidal two-planet belts, with the vector-radii of the perihelion planets in each of the perihelion or infra-asteroidal belts, as is shown in the following table. The tabular unit is Sun's radius:

| | A | | | B | (A-B) ÷ B |
|---|---|---|---|---|---|
| ♄ $\times \pi^1$ | 6438.75 | ♆ | *mean* | 6453.06 | —.0022 |
| ♄ $\times \pi^0$ | 2049.51 | ♄ | " | 2049.51 | |
| ♄ $\times \pi^{-1}$ | 652.38 | ⚹ | " | 652.38 | |
| ♄ $\times \pi^{-2}$ | 207.66 | ⊕ | *perihelion* | 207.58 | +.0004 |
| ♄ $\times \pi^{-3}$ | 66.10 | ☿ | " | 68.48 | —.0361 |
| ⊕ $\times \pi^{-1}$ | 68.39 | ☿ | " | 68.48 | —.0013 |

The close accordance between the deviation of ♄ $\times \pi^{-3}$ and ⊕'s mean eccentricity, connects the supra-asteroidal with the infra-asteroidal planets, in a manner which is still further illustrated by the Neptunian, Jovian, and Telluric harmonic series of planetary positions*.

In the harmonic series of differences between perihelion nucleal pendulums $(\frac{56}{25})$, $(\frac{56}{24})$, $(\frac{56}{23})$, $(\frac{56}{22})$, $(\frac{56}{21})$† the inter-telluric terms were $\frac{1}{2}$ Venus

* *Ante*, xiii, 239. † *Ante*, xiv, 628.

and $\frac{1}{2}$ Mercury. If we take seven geometrical means between $\frac{1}{2}$ Mercury's, and Neptune's, mean radius-vector, we find the following accordances :

| | Theoretical (T). | Observed (O). | (T-O) ÷ O. |
|---|---|---|---|
| $\frac{1}{2}$ ☿$_2$ | .19355 | .19355 | |
| $\frac{1}{2}$ ♀$_2$ | .36362 | .36167 | +.0054 |
| $\frac{2}{3}$ ⊕$_3$ | .68312 | .68925 | —.0090 |
| $\frac{3}{4}$ ♂$_2$ | 1.28337 | 1.23312 | +.0407 |
| $\frac{4}{5}$ * | 2.41103 | | |
| $\frac{5}{6}$ ♃$_3$ | 4.52954 | 4.52279 | +.0015 |
| $\frac{6}{7}$ ♄$_3$ | 8.50918 | 8.57149 | —.0073 |
| $\frac{7}{8}$ ♅$_1$ | 15.98670 | 16.03259 | —.0029 |
| $\frac{8}{8}$ ♆$_2$ | 30.0334 | 30.0334 | |

The geometric ratio of the theoretical column is 1.879, or almost precisely the sum of the co-efficients of the Urano-Neptunian belt ($\frac{7}{8} + \frac{8}{8}$). It will be observed that the theoretical co-efficients ($\frac{1}{2}$ ♀, $\frac{2}{3}$ ⊕ . . . $\frac{7}{8}$ ♅) are the same as appear in the inter-planetary abscissas of my Centaurus-Heliacal parabola.* The collisions of particles, in their approach to the focus of a paraboloid, would naturally convert parabolic into elliptical orbits; and particles falling towards a cosmic focus from a distance $nr$ would acquire the dissociative velocity (relatively to the Sun) $\sqrt{2gr}$, at $\frac{nr}{n+1}$ from the focus. By giving to $n$, successive values in arithmetical progression, we form the arithmetico-harmonic series, $\frac{1}{2}$ $\frac{2}{3}$ $\frac{3}{4}$ $\frac{4}{5}$ $\frac{5}{6}$ $\frac{6}{7}$ $\frac{7}{8}$, which constitute the peculiar sequence of co-efficients, both in the foregoing geometric series and in the abscissas of the primitive parabola.

The bases of the principal planetary harmonies that have been hitherto published, are :—Peirce (phyllotactic), the time of orbital revolution, $t$; Bode, and Alexander, the orbital radius vector, or the radius of possible solar-nebular atmosphere, $r$; my own (harmonic), the nucleal radius, $\rho$. Their common relations may be thus shown :—

Peirce, $t \propto t^{1} \propto r^{\frac{3}{2}} \propto \rho^{2}$

Bode, Alexander, $r \propto t^{\frac{2}{3}} \propto r^{1} \propto \rho^{\frac{4}{3}}$

Chase, $\rho \propto t^{\frac{1}{2}} \propto r^{\frac{3}{4}} \propto \rho^{1}$

The Saturnian relations of inertia seem to have established the Bode series. For if we take as our unit, $\rho_0 = 20.58$ solar radii, ($\rho_0$ being the nucleus of a nebulous sun which would rotate synchronously with Saturn's orbital revolution), we obtain the following values :—

| Bode (T). | | Actual (O). | | (T-O) ÷ O. |
|---|---|---|---|---|
| $1 = \rho_0$ | 20.58 | ♄ ☉ *nucleus* | 20.58 | |
| 4 | 82.31 | ☿$_2$ | 83.36 | —.013 |
| 7 | 144.04 | ♀$_1$ | 149.93 | —.041 |
| 10 | 205.78 | ⊕$_1$ | 207.58 | —.009 |
| 16 | 329.23 | ♂$_2$ | 329.74 | —.002 |

* *Ante*, xii, 523-1.

| | Bode (T). | | Actual (O). | (T-O) ÷ O. |
|---|---|---|---|---|
| 28 | 726.16 | Asteroidal | | |
| 52 | 1070.06 | $♃_1$ | 1069.62 | +.000 |
| 100 | 2057.78 | $♄_2$ | 2049.51 | +.004 |
| 196 | 4033.25 | $♅_2$ | 4121.76 | —.022 |
| [292 | 6008.71] | $♆_1$ | 6388.25 | —.063 |
| 388 | | | | |

It will be observed that I have interpolated the Neptunian term, but this modification of the Bodeian law, as I have, in part, previously stated, increases its harmony, by giving three equal differences at each extremity of the series, by placing Earth's perihelion in a geometrical mean position between the ♄ ☉ nucleus and its limit of possible atmosphere, and by marking centres of linear oscillation of successive pendulums.

After the hypothetical detachment of the several two-planet belts, and their independent revolution preparatory to cosmical division, the harmonic should replace the geometric ratios. In order to remove the influence of the theoretical planetary pendulum unit ($\frac{1}{8}$☉$r$) and the slight uncertainty as to the precise period of solar rotation, let us examine the *ratios* of the several planetary rotation- (or nucleal-) radii, and the consequent harmonic differences, according to the above equation of variability, $\rho \propto t^{\frac{1}{2}}$.

| | $t$ | $\rho$ |
|---|---|---|
| ☿ | .2408 | .491 |
| ♀ | .6152 | .784 |
| ⊕ | 1.0000 | 1.000 |
| ♂ | 1.8808 | 1.374 |
| ♃ | 11.8618 | 3.444 |
| ♄ | 29.4565 | 5.427 |
| ♅ | 84.0190 | 9.166 |
| ♆ | 164.7791 | 12.837 |

☿ ÷ ♀ = 5 ÷ 7.99 ♃ ÷ ♄ = 5 ÷ 7.88

⊕ ÷ ♂ = 5 ÷ 6.86 ♅ ÷ ♆ = 5 ÷ 7.

$$♅ - ♃ = 5.722 \quad \frac{12.837}{5.722} = \frac{56}{24.96}$$

$$♃ - ⊕ = 2.444 \quad \frac{5.722}{2.444} = \frac{56}{23.92}$$

$$⊕ - ☉ = 1.000 \quad \frac{2.444}{1.000} = \frac{56}{22.91}$$

$$☉ - \tfrac{1}{2}♀ = .392 \quad \frac{1.000}{.392} = \frac{56}{21.95}$$

$$\tfrac{1}{2}♀ - \tfrac{1}{2}☿ = .147 \quad \frac{.392}{.147} = \frac{56}{21}$$

By comparing the radii of æthereal nebulosity; of synchronous central and circular oscillation ($2r : r$); of incipient aggregation, or constant pressure ($1.4232r$); and of nebular rupture $\left(\frac{nr}{n+1}\right)$, we find the following accordances:

1. An æthereal atmosphere, rotating with planetary velocity at Sun's present surface, would have the equatorial velocity of light at $688.33r$.

| | | |
|---|---|---|
| 3 × 688.33 | 2064.99 $♄_2$ | 2049.51 |
| 6 × " | 4129.98 $♅_2$ | 4121.86 |
| 9 × " | 6194.97 $♆_1$ | 6388.25 |

2. If the radial oscillation and the radius of nebular rupture are specially regarded, $r = \frac{1}{2}$ radius vector; $n = 2$; $\frac{nr}{n+1} = \frac{2r}{3} = \frac{1}{3}$ radius vector.

| | | | | |
|---|---|---|---|---|
| $\frac{1}{3}$ ♃ | 1.6594 | ♂$_3$ | 1.6444 |
| $\frac{1}{3}$ $\Psi_2$ | 10.0113 | ♄$_3$ | 10.0000 |
| $\frac{1}{3}$ ⊕$_2$ | .3333 | ☿$_1$ | .3187 |

3. Substituting the orbital radius for the radius of linear oscillation, we have $\frac{nr}{n+1} = \frac{2}{3}$.

| | | | |
|---|---|---|---|
| $\frac{2}{3}$ $\Psi$ | 20.0226 | ♅$_3$ | 20.0442 |
| $\frac{2}{3}$ ⊕$_3$ | .6763 | ♀$_1$ | .6978 |

4. Substituting the radius of incipient aggregation and its corresponding radius of linear oscillation, we have

$$\frac{nr}{n+1} = \frac{1.4232\, r}{2}; \quad \therefore\ n = 2.467$$

| | | | |
|---|---|---|---|
| $2\,n$ | 4.934 | ♃$_1$ | 4.978 |

The combined influences of Jupiter and Earth over the asteroidal belt, especially as shown in the second and fourth accordances; the tendency of their mean radial velocities (at $1.4232\, r$) and the limiting satellite velocities, to equality at Sun's present limiting planetary velocity; the indications of uniform primitive velocity, furnished by the general predominance of geometrical ratios and the introduction of harmonic values in the minute details; the *a priori* probability of such primitive uniformity; the relations of mass and position to orbital times, as well as to atmospheric and nuclear-nebular radii ($t$, $r$, and $\rho$); all point to originating undulations, propagated, as inferred from the ultimate limit of equality towards which the parabolic cometary and mean radial centrifugal velocities both tend, with the velocity of light.

La Place (*Mécanique Celeste*, II, viii, 65–69; VI, ii, 12–16; etc.) investigated a number of inequalities depending on the squares and products of the disturbing forces. In his discussions of the Jovian and Saturnian systems he introduced terms containing the 3d and 5th dimensions of the eccentricities and inclinations. The closeness of the agreements here presented may, perhaps, lead to important considerations involving still higher powers.

If we substitute for the theoretical primitive exponential ratios (1, 1+2, 1+2+3), the present actual vector radii, ($\alpha = \Psi_2$; $\beta = $ ♅$_2$; $\gamma = $ ♄$_2$; $\delta = $ ♃$_2$), we find an equation for Saturn's mean perihelion :—

$$\Psi_2^{\,\delta} \times \text{♅}_2^{\,\beta-\delta} \times \text{♃}_2^{\,\alpha} = \text{♄}_1^{\,\alpha+\beta} \qquad (1)$$

If $\alpha$, $\beta$, $\delta$, represent the mean aphelion vector-radii, we find an equation for Saturn's mean distance :—

$$\Psi_3^{\,\delta} \times \text{♅}_3^{\,\beta-\delta} \times \text{♃}_3^{\,\alpha} = \text{♄}_2^{\,\alpha+\beta} \qquad (2)$$

If we take powers of the masses, instead of powers of the vector-radii, equation (2) gives two values for Saturn's mass, according as we use Newcomb's greatest value of Neptune's mass, $\left(\frac{1}{19380}\right)$, deduced from its satellite (3)

or the least, $\left(\frac{1}{19700}\right)$, deduced from perturbations of ♅ (4)

These equations are immediately suggestive of the numerous familiar equations between the *sums* of periodic times. The substitution of *products* for sums, and powers for products, indicates the early organizing activity of constant forces, acting with reference to given centres, in elastic media.

The solution of equations (1) to (4) is as follows :—

(1)

| | |
|---|---|
| Log. $30.03386^{5.202798}$ | 7.687712 |
| " $19.183581^{19.183581-5.202798}$ | 17.936362 |
| " $5.202798^{30.03386}$ | 21.511361 |
| | 47.135435 |
| " $9.077645^{30.03386+19.183581}$ | 47.148979 |

$$.013544 \div 49.217441 = .000275 = \log.\ 1.00051$$

(2)

| | |
|---|---|
| Log. $30.33551^{5.427351}$ | 8.043068 |
| " $20.044183^{20.044183-5.427351}$ | 19.030955 |
| " $5.427351^{30.33551}$ | 22.284102 |
| | 49.358125 |
| " $9.538852^{30.3351+20.044183}$ | 49.346714 |

$$.011411 \div 50.379693 = .000227 = \log.\ 1.00050$$

(3), (4)

If log. ⊙ = 10, the logs. of the assumed masses are :—

| | |
|---|---|
| ♆ (Newcomb, from satellite) | 5.712646 |
| ♆ ( " " perturbations) | 5.705534 |
| ♅ ( " ) | 5.645892 |
| ♄ (Bessel) | 6.455734 |
| ♃ ( " ) | 6.979689 |

Substituting these logs. for the aphelion logs. in equation (2), we get for log. ♄, by using for log. ♆

| | | |
|---|---|---|
| Satellite value . | (3) | 6.458198 |
| Perturbation value | (4) | 6.456439 |

$$6.458198 - 6.455734 = .002464 = \log.\ 1.0057$$
$$6.456439 - 6.455734 = .000705 = \log.\ 1.0016$$